

WOODIN'S SURGERY METHOD

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ABSTRACT. In this short paper we give an overview of Woodin's surgery method.

1. SURGERY METHOD FOR STRONG CARDINALS

In this section we present an abstract version of Woodin's surgery method for strong cardinals.

Theorem 1.1. ([3]) *Let $j : M \rightarrow N$ be an elementary embedding with $\kappa = \text{crit}(j)$, where κ is inaccessible in M , and $N = \{j(F)(a) : F \in M, F : [\kappa]^{<\omega} \rightarrow M \text{ and } a \in [\lambda]^{<\omega}\}$. Let $\mathbb{P} = \text{Add}(\kappa, \nu)_M$, where ν is a cardinal in M and suppose that $j \restriction \nu \in M$. Let G be \mathbb{P} -generic over M , and suppose that there exists H such that:*

- (1) $N \subseteq M[G]$,
- (2) $M[G] \models N^\kappa \subseteq N$,
- (3) H is $j(\mathbb{P})$ -generic over $M[G]$.

Then there exists $H^ \in M[G][H]$ such that H^* is $j(\mathbb{P})$ -generic over N and $j[G] \subseteq H^*$.*

We will present two proofs of the above theorem. The first one, essentially due to Woodin, is taken from [1].

First proof. Let

$$g = \bigcup G : \nu \rightarrow 2,$$

$$h = \bigcup H : j(\nu) \rightarrow 2.$$

Define $h^* : j(\nu) \rightarrow 2$ by

$$h^*(\beta) = \begin{cases} g(\alpha) & \text{if } \beta = j(\alpha), \\ h(\beta) & \text{Otherwise.} \end{cases}$$

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Let H^* be the filter generated by h^* . Note that $H^* = \{p^* : p \in H\}$, where for each $p \in j(\mathbb{P})$, p^* is defined by

- $\text{dom}(p^*) = \text{dom}(p)$,
- p^* is defined by

$$p^*(\beta) = \begin{cases} g(\alpha) & \text{if } \beta = j(\alpha), \\ p(\beta) & \text{Otherwise.} \end{cases}$$

Let's first show that H^* is well-defined.

Lemma 1.2. $p \in j(\mathbb{P}) \Rightarrow p^* \in j(\mathbb{P})$.

Proof. It suffices to show that $p^* \in N$. But clearly $p^* \in M[G]$, so by clause (2) of the theorem, it suffices to show that $X(p, p^*) = \{z : p(z) \neq p^*(z)\}$ has size $\leq \kappa$. We have $X(p, p^*) \subseteq \text{dom}(p) \cap j[\nu]$, so it suffices to show that the later set has size at most κ . Let $p = j(F)(a)$, where $a \in [\lambda]^{<\omega}$, $F \in M$, $F : [\kappa]^{|a|} \rightarrow M$. We may further suppose that $\forall x \in [\kappa]^{|a|}, F(x) \in \mathbb{P}$. Then

$$\alpha < \nu, j(\alpha) \in \text{dom}(p) \Rightarrow \exists x, \alpha \in \text{dom}(F(x)).$$

So if $X = \bigcup \{\text{dom}(F(x)) : x \in [\kappa]^{|a|}\}$, then $X \in M$ and $M[G] \models " |X| \leq \kappa \text{ and } \text{dom}(p) \cap j[\nu] \subseteq j[X]"$. The result follows. \square

It is easily seen that H^* is a filter on $j(\mathbb{P})$.

Lemma 1.3. H^* is $j(\mathbb{P})$ -generic over N .

Proof. Let $D \in N$ be dense open in $j(\mathbb{P})$. Define an equivalence relation on $j(\mathbb{P})$ by

$$p \sim q \Leftrightarrow \text{dom}(p) = \text{dom}(q) \text{ and } |\{z : p(z) \neq q(z)\}| \leq \kappa.$$

Let $E = \{q \in j(\mathbb{P}) : \forall p, p \sim q \Rightarrow p \in D\}$. We show that E is dense in $j(\mathbb{P})$. First we prove the following.

Lemma 1.4. If $p \in j(\mathbb{P})$, then there is $q \leq p$ such that

$$\forall r, r \sim q \Rightarrow r \cup (q \setminus p) \in D.$$

Proof. Let $\langle X_\alpha : \alpha < \mu \rangle, \mu < j(\kappa)$ be an enumeration of $\{X \subseteq \text{dom}(p) : |X| \leq \kappa\}$. Define by induction a decreasing sequence $\langle p_\alpha : \alpha \leq \mu \rangle$ of conditions as follows:

- $\alpha = 0$: Let $p_0 = p$,
- $\alpha = \beta + 1$: Suppose p_β is defined. Let

$$q(z) = \begin{cases} p_\beta(z) & \text{if } z \in \text{dom}(p_\alpha) \setminus X_\alpha, \\ 1 - p_\beta(z) & \text{Otherwise.} \end{cases}$$

Since D is dense, we can find $\bar{q} \in D$ such that $\bar{q} \leq q$. Set $p_\alpha = p_\beta \cup (\bar{q} \setminus q)$.

- α is a limit ordinal: Let $p_\alpha = \bigcup_{\beta < \alpha} p_\beta$.

Then $q = p_\mu$ is as required. \square

Lemma 1.5. *E is dense in $j(\mathbb{P})$.*

Proof. Let $p \in j(\mathbb{P})$. Using the above claim κ^+ -times, we can produce a decreasing sequence $\langle p_\alpha : \alpha < \kappa^+ \rangle$ of conditions extending p such that for any $\alpha < \kappa^+$ if $r \sim p_\alpha$, then $r \cup (p_{\alpha+1} \setminus p_\alpha) \in D$. Let $q = \bigcup_{\alpha < \kappa^+} p_\alpha$. Then $q \leq p$ and $q \in E$. To see this just note that if $r \sim q$, then for some $\alpha < \kappa^+$, $X(r, q) \subseteq \text{dom}(p_\alpha)$, so $X(r, q) = X(r, p_\alpha)$. \square

Let $p \in H \cap E$. Then $p^* \sim p$, so $p^* \in H^* \cap D$. The theorem follows. \square

Second proof. Let H^* be as defined above. We show that it is $j(\mathbb{P})$ -generic over N . Thus let $A \in N$ be a maximal antichain of $j(\mathbb{P})$. Then $|A| \leq j(\kappa)$. Set

$$S = \bigcup \{ \text{dom}(p) : p \in A \}.$$

then $N \models "S \subseteq j(\nu) \text{ and } |S| \leq j(\kappa)"$. Let $S = j(F)(a)$, where $a \in [\lambda]^{<\omega}$, $F \in M$, $F : [\kappa]^{|a|} \rightarrow M$. We may further suppose that $M \models " \text{For each } x \in \text{dom}(F), f(x) \subseteq \nu \text{ and } |f(x)| \leq \kappa "$. Set $T = \bigcup \{ f(x) : x \in [\kappa]^{|a|} \}$. Then $T \in M$ and $M \models "T \subseteq \nu \text{ and } |T| \leq \kappa"$. It is easily seen that

$$M[G] \models "S \cap j[\nu] \subseteq j[T]"$$

Hence by clause (2), $S \cap j[\nu] \in N$. Let $X_0 = S \cap j[\nu]$ and $X_1 = j(\nu) \setminus X_0$. Also set $\mathbb{P}_i = \{ p \in j(\mathbb{P}) : \text{dom}(p) \subseteq X_i \}, i = 0, 1$. Then we have a natural forcing isomorphism

$$\pi : j(\mathbb{P}) \rightarrow \mathbb{P}_0 \times \mathbb{P}_1,$$

given by

$$\pi(p) = \langle p \restriction X_0, p \restriction X_1 \rangle.$$

Note that $H^* \restriction X_0 \in \mathbb{P}_0$. Set

$$A_1 = \{p \restriction X_1 : p \in A \text{ and } p \text{ is compatible with } h^* \restriction X_0\}.$$

The following lemma can be proved quite easily.

Lemma 1.6. *A_1 is a maximal antichain in \mathbb{P}_1 .*

On the other hand $H_1 = \{p \restriction X_1 : p \in H\}$ is \mathbb{P}_1 -generic, so $H_1 \cap A_1 \neq \emptyset$. Let $p \in A$ be such that p is compatible with $h^* \restriction X_0$ and $p \restriction X_1 \in H_1 \cap A_1$. But then $p \in H^* \cap A$, and hence $H^* \cap A \neq \emptyset$. The theorem follows. \square

2. SURGERY METHOD FOR SUPERCOMPACT CARDINALS

In this section we prove the following theorem, which is an analogue of Theorem 1.1 for supercompact cardinals.

Theorem 2.1. *Let $j : M \rightarrow N$ be an elementary embedding with $\text{crit}(j) = \kappa$, where κ is inaccessible in M , and $N = \{j(F)(j[\lambda]) : F \in M, F : P_\kappa(\lambda) \rightarrow M\}$. Let $\mathbb{P} = \text{Add}(\kappa, \nu)_M$, where ν is a cardinal in M and suppose that $j \restriction \nu \in M$. Let G be \mathbb{P} -generic over M , and suppose that there exists H such that:*

- (1) $N \subseteq M[G]$,
- (2) $M[G] \models N^\lambda \subseteq N$,
- (3) H is $j(\mathbb{P})$ -generic over $M[G]$.

Then there exists $H^ \in M[G][H]$ such that H^* is $j(\mathbb{P})$ -generic over N and $j[G] \subseteq H^*$.*

Proof. Let g, H and H^* be defined as before. We show that H^* is as required.

Let $A \in N$ be a maximal antichain of $j(\mathbb{P})$. Then $|A| \leq j(\kappa)$. Set

$$S = \bigcup \{\text{dom}(p) : p \in A\}.$$

Then $N \models "S \subseteq j(\nu) \text{ and } |S| \leq j(\kappa)"$. Let $S = j(F)(j[\lambda])$, where $F \in M, F : P_\kappa(\lambda) \rightarrow M$. We may further suppose that $M \models " \text{For each } x \in \text{dom}(F), f(x) \subseteq \nu \text{ and } |f(x)| \leq \kappa "$. Set $T = \bigcup \{f(x) : x \in P_\kappa(\lambda)\}$. Then $T \in M$ and $M \models "T \subseteq \nu \text{ and } |T| \leq \lambda"$. It is easily seen that

$$M[G] \models "S \cap j[\nu] \subseteq j[T]"$$

Thus by clause (2), $S \cap j[\nu] \in N$. The rest of the argument is as before. \square

As an application of the above theorem, we give a proof of the following theorem (compare with [2], Section 13).

Theorem 2.2. *Assume GCH holds and κ is κ^+ -supercompact. Then there is a generic extension in which κ remains κ^+ -supercompact and $2^\kappa = \kappa^{++}$.*

Proof. Let $\mathbb{P} = \mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^{++})$ be the reverse Easton iteration for adding α^{++} -many new Cohen subsets of α , using $\text{Add}(\alpha, \alpha^{++})$, for all inaccessible cardinals $\alpha \leq \kappa$, and let $G * g$ be \mathbb{P} -generic over V . Let $j : V \rightarrow M \simeq \text{Ult}(V, U)$, where U is a normal measure on $P_\kappa(\kappa^+)$, so that $M = \{j(F)(j[\kappa^+]) : F \in V, F : P_\kappa(\kappa^+) \rightarrow V\}$. Also let $j(\mathbb{P}) = \mathbb{P} * \mathbb{R} * \text{Add}(j(\kappa), j(\kappa^{++}))$.

By standard forcing arguments we can find $H \in V[G * g]$ which is $j(\mathbb{P}_\kappa) = \mathbb{P} * \mathbb{R}$ -generic over M , and since $j[G] \subseteq G * g * H$, We can lift j to some $j : V[G] \rightarrow M[G * g * H]$.

Let h be $\text{Add}(j(\kappa), j(\kappa^{++}))_{M[G * g * H]}$ -generic over $V[G * g * H]$. Applying Theorem 2.1, there exists h^* such that we have the lifting $j^* : V[G * g] \rightarrow M[G * g * H * h^*]$. Working in $V[G * g * H * h]$, define U^* on $P_\kappa(\kappa^+)$ by

$$X \in U^* \Leftrightarrow j[\kappa^+] \in j^*(X).$$

Note that

$$(*) \quad V^\mathbb{P} \models \text{"}\mathbb{R} * \text{Add}(j(\kappa), j(\kappa^{++})) \text{ is } \leq \kappa^+ \text{-closed"}. \quad$$

Using $(*)$, $U^* \in V[G * g]$, and $V[G * g] \models \text{"}U^* \text{ is a normal measure on } P_\kappa(\kappa^+) \text{"}$. The theorem follows. \square

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